Higher-Order Probability Theory on Interval Domains

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Probabilistic PLs So Far

1979	Kozen	Semantics of Probabilistic Programs
1999	Panangaden	The Category of Markov kernels
2008	Park et al.	A Probabilistic Language Based on Sampling Functions
2011	Danos et al.	Probabilistic Coherence Spaces as a Model of Higher-order Probabilistic Computation
2014	Ehrhard et al.	Full Abstraction for Probabilistic PCF
2016	Borgstrom et al.	A Lambda-Calculus Foundation for Universal Probabilistic Programming
2016	Staton et al.	Semantics for Probabilistic Programming: Higher-order Functions, Continuous Distributions, and Soft Constraints
2017	Culpepper et al.	Contextual Equivalence for Probabilistic Programs with Continuous Random Variables and Scoring
2017	Staton	Commutative Semantics for Probabilistic Programming
2017	Heunen et al.	A Convenient Category for Higher-order Probability Theory
2018	Ehrhard et al.	Measurable Cones and Stable, Measurable Functions
2018	Wand et al.	Contextual Equivalence for a Probabilistic Language with Continuous Random Variables and Recursion
2019	Vákár et al.	A Domain Theory for Statistical Probabilistic Programming

PL Untyped, simply-typed or recursively-typed; CBN or CBV;
First or higher-order; Recursion;

Prob Discrete and/or continuous; Hard and/or soft constraints

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Studying Prob PCF (Vákár, Kammar, and Staton 2019)

Syntax	CBV Prob PCF (pPCF)
Denotational Semantics	Interpretation of type $[\![\Gamma]\!]$ and Interpretation of term $[\![s]\!] : [\![\Gamma]\!] \to T[\![\sigma]\!]$ using ω**Qbs** and Integration Monad T
Operational Semantics	Behaviour of term $\langle\!\langle s \rangle\!\rangle$ using kernels $\longrightarrow_n : \Lambda^{\vdash\sigma} \to T\Lambda^{\vdash\sigma}$
Contextual Equivalence	$s \sim_{ctx} t$ if for any program context $C[-]$, $\langle\!\langle C[s] \rangle\!\rangle = \langle\!\langle C[t] \rangle\!\rangle$
Correctness ✓	$s \longrightarrow_n \int \alpha$ implies $[\![s]\!] = \lambda f. \int_{\Lambda^{\vdash\sigma}} [\![t]\!] f \; (\alpha_* \mathrm{Leb})(dt)$
Main Lemma ✓	$[\![s]\!] = \langle\!\langle s \rangle\!\rangle$
Adequacy ✓	$[\![s]\!] = [\![t]\!]$ implies $s \sim_{ctx} t$
Full Abstraction ?	$[\![s]\!] = [\![t]\!]$ if and only if $s \sim_{ctx} t$

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Probabilistic coherence spaces is a fully abstract model for Prob PCF (Ehrhard, Tasson, and Pagani 2014)

Is $\omega\mathbf{Qbs}$ a fully abstract model for pPCF?

Is ω**Qbs** a fully abstract model for pPCF?

Cartesian closed category containing **Sbs** as a full sub-category

ωQbs (Vákár, Kammar, and Staton 2019)

Object: a triple $(X, M_X, \sqsubseteq)$ where

- (X, M_X) is a **qbs** (Heunen et al. 2017);
- $(X, \sqsubseteq)$ is an ω-**cpo**

such that M_X is closed under point-wise lubs of ω-chains.

Arrow: f that is a **qbs-morphism** and is **Scott-continuous**.

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pPCF (Vákár, Kammar, and Staton 2019)

$$\text{Types} \quad \sigma, \tau ::= \mathfrak{R} \mid \sigma \Rightarrow \tau$$

$$\text{Terms} \quad s, t ::= x \mid r \mid \lambda x.s \mid s\,t \mid \text{if } b \text{ then } s \text{ else } t \mid \text{Y}s \mid \underline{f}(s_1, \ldots, s_n) \mid \text{score}(s) \mid \text{sample}$$

where $r \in \mathbb{R}$ and $f \in \omega\mathbf{Qbs}(\mathbb{R}^n, \mathbb{R})$.

Proving Full Abstraction

denotational semantics

operational semantics

Full Abstraction

$[\![s]\!] = [\![t]\!]$ if and only if $s \sim_{ctx} t$, i.e. $\forall C[-]$, $\langle\!\langle C[s] \rangle\!\rangle = \langle\!\langle C[t] \rangle\!\rangle$.

($\Rightarrow$) Adequacy (Vákár, Kammar, and Staton 2019)

($\Leftarrow$) By contraposition.

1. Assume $[\![s]\!] \neq [\![t]\!]$.
2. Construct an element d such that $d \bullet [\![s]\!] \neq d \bullet [\![t]\!]$.
3. Define d using a term f in Λ, hence $[\![(\lambda x.f\, x)\, s]\!] \neq [\![(\lambda x.f\, x)\, t]\!]$.
4. By Main Lemma, $\langle\!\langle C[s] \rangle\!\rangle \neq \langle\!\langle C[t] \rangle\!\rangle$ where $C[-] \equiv (\lambda x.f\, x)\, [-]$ and hence $s \not\sim_{ctx} t$.

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monadic application

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$[\![u]\!] = \langle\!\langle u \rangle\!\rangle \; \forall u \in \Lambda$

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- For each $[\![\sigma]\!]$, identify a subset $B \subseteq [\![\sigma]\!]$ such that elements in B can distinguish distinct elements.
- Show that every element in B is definable in Λ, i.e. $\forall b \in B \;.\; \exists s \in \Lambda \;.\; [\![s]\!] = b$.

Domain Theoretical Framework

x is a lot simpler than y

ω-Continuous Domains (Plotkin 1977)

- $x \ll y$ if for any ω-chain (z_n) such that $y \sqsubseteq \bigsqcup z_n$, $x \sqsubseteq z_n$ for some $n \in \mathbb{N}$.

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Two Steps to Full Abstraction

1. Show that for any type σ, $[\![\sigma]\!]$ is ω-continuous.
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Two Steps to Full Abstraction

1. Show that for any type σ, $[\![\sigma]\!]$ is ω-continuous.
   1.1 Show that $[\![\mathfrak{R}]\!]$ is ω-continuous.
   1.2 Find conditions on ω-continuous ω-qbses X and Y such that $\omega\mathbf{Qbs}(X, Y)$ is ω-continuous.
   1.3 Find conditions on ω-continuous ω-qbs X such that TX is ω-continuous.
2. Show that all basis elements of $[\![\sigma]\!]$ are definable.

Step 1.1: Show that $[\![\mathfrak{R}]\!]$ is ω-continuous

$[\![\mathfrak{R}]\!] := (\mathbb{R}, \textbf{Meas}(\mathbb{R}, \mathbb{R}), =)$ is **not** ω-continuous.

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Interval Domain $(\mathbf{I}\mathbb{R}, \sqsubseteq)$ (Dana Scott)

$\mathbf{I}\mathbb{R}$ is the set of all closed intervals (partial real numbers) on $\mathbb{R}$.

$\sqsubseteq$ is a partial order where $\mathbf{r} \sqsubseteq \mathbf{q}$ iff $\mathbf{r} \supseteq \mathbf{q}$. e.g. $[1, 4] \sqsubseteq [2, 3.4]$.

$(\mathbf{I}\mathbb{R}, \sqsubseteq)$ is ω-**continuous** with basis $B(\mathbf{I}\mathbb{R}) = \{[q_1, q_2] \mid q_1, q_2 \in \mathbb{Q}\}$.

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Interval ω-qbs $(\mathbf{I}\mathbb{R}, M_{\mathbf{I}\mathbb{R}}, \sqsubseteq)$

$M_{\mathbf{I}\mathbb{R}}$ is the smallest set of random elements that

- contains all constant functions and $\lambda r.[r, r]$, and
- closed under qbs-axioms and pointwise lub of ω-chains.

If $f : \mathbb{R} \to \mathbb{R}$ is measurable, then $\lambda r.[f(r), f(r)]$ is in $M_{\mathbf{I}\mathbb{R}}$.

Step 1.1: Show that $[\![\mathfrak{R}]\!]$ is ω-continuous

- Then we have $[\![\mathfrak{R}]\!] := (\mathbf{I}\mathbb{R}, M_{\mathbf{I}\mathbb{R}}, \sqsubseteq)$, which is ω-continuous.

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- But $[2, 4]$ is not definable in pPCF.

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IpPCF = PCF + Partial Real Numbers (Escardó 1996) + sample + score(c)

Types $\quad \sigma, \tau ::= \mathfrak{R} \mid \mathfrak{B} \mid \sigma \Rightarrow \tau$

Terms $\quad s, t ::= x \mid [r_1, r_2] \mid \mathrm{tt} \mid \mathrm{ff} \mid \lambda x.s \mid s\,t \mid \mathrm{if}\ b\ \mathrm{then}\ s\ \mathrm{else}\ t \mid \mathrm{Y}s$
$\qquad \mid \underline{f}(s_1, \ldots, s_n) \mid \mathrm{score}(s) \mid \mathrm{sample}$

where $f \in \omega\mathbf{Qbs}(\mathbf{I}\mathbb{R}^n, \mathbf{I}\mathbb{R}) \cup \omega\mathbf{Qbs}(\mathbf{I}\mathbb{R}^n, \{\mathrm{tt}, \mathrm{ff}\}_\perp)$

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where $f \in \omega\mathbf{Qbs}(\mathbf{I}\mathbb{R}^n, \mathbf{I}\mathbb{R}) \cup \omega\mathbf{Qbs}(\mathbf{I}\mathbb{R}^n, \{\mathsf{tt}, \mathsf{ff}\}_\bot)$

$\mathsf{cons}_{-}(-), \mathsf{tail}_{-}(-)$ and $(-) <_\bot r$ can be expressed via $\underline{f}$

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Adequacy

ω**Qbs** is an adequate model for IpPCF using the integration monad T.

Step 1.2: Conditions on ω-continuous ω-qbs such that the **exponential** is also ω-continuous

For any ω-continuous ω-cpos X, Y where Y is b-c and pointed, $\omega\mathbf{Cpo}(X, Y)$ is b-c, pointed and ω-continuous.

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Definitions on ω-qbs

Let $(X, M_X, \sqsubseteq)$ be an ω-qbs. It is

- **pointed** if $(X, \sqsubseteq)$ is pointed;
- **bounded-complete** if $(X, \sqsubseteq)$ and M_X are bounded-complete;
- **stable** if $(X, \sqsubseteq)$ is stable, i.e. $U \ll V$ and $U \ll V'$ implies $U \ll V \cap V'$ for all $U, V, V' \in X_\sigma$;
- **tree-like** if $(X, \sqsubseteq)$ is tree-like, i.e. $\downarrow x$ are chains;
- ω-**continuous** if $(X, \sqsubseteq)$ is ω-continuous;
- **upper-measurable** if all open sets in X_σ is measurable;
- **sharp** if it is ω-continuous and upper-measurable.

Using results in Erker, Escardó, and Keimel 1998; Goubault-Larrecq 2013, For any sharp ω-qbses X and Y where Y is bounded-complete and pointed and either X is stable or Y is tree-like, the exponential $\omega\mathbf{Qbs}(X, Y)$ is bounded-complete, pointed and sharp (hence stable).

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b-c and ω-continuous implies stability

Step 1.3: Condition on ω-continuous ω-qbs X such that TX is ω-continuous.

- Integration monad TX is the ω-chain closure of the image of $\int := \lambda \alpha f . \int_{\alpha^{-1}(X)} f \circ \alpha \, d\mathrm{Leb}$ (Vákár, Kammar, and Staton 2019).

Randomisation Lemma: every s-finite measure can be defined as the push-forward of a random element along Leb, $\alpha_* \mathrm{Leb}$

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Interval Integration Monad $\mathbf{I}T$

$$\begin{aligned} \oint : \quad \mathbf{I}\mathbb{R} \Rightarrow X_\perp &\longrightarrow (X \Rightarrow \mathbb{L}_+) \Rightarrow \mathbb{L}_+ \\ \alpha &\longmapsto \int (\alpha \circ \lambda r.[r, r]) \end{aligned}$$

$\mathbf{I}TX$ is the ω-chain closure of the image of $\oint$

$M_{\mathbf{I}TX}$ is the ω-chain closure $\{\oint \circ \alpha \mid \alpha \in \mathbb{R} \Rightarrow (\mathbf{I}\mathbb{R} \Rightarrow X_\perp)\}$

- $\mathbf{I}TX \subseteq TX$.
- $\mathbf{I}TX$ is an ω-**qbs**.
- $\mathbf{I}T$ is a **commutative sub-monad** of J.
- For any sharp ω-qbs X, $\mathbf{I}TX$ is bounded-complete, pointed and **sharp**.

Interval Integration Monad $\mathbf{I}T$

Example: Dirac distribution $\mathbf{I}\delta_x : \mathbf{I}\mathbb{R} \to X_\perp$

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Interval Integration Monad $\mathbf{I}T$

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$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases} \qquad \mathbf{I}\delta_x := \lambda[r_1, r_2].\begin{cases} x & \text{if } r_1, r_2 \in (0,1), \\ \perp & \text{otherwise.} \end{cases}$$

$\mathbf{I}\delta_x$ is a qbs-morphism and Scott-continuous. Moreover,

$$\oint \mathbf{I}\delta_x = \int (\mathbf{I}\delta_x \circ \lambda r.[r, r]) = \int (\lambda r.\begin{cases} x & \text{if } r \in (0,1), \\ \perp & \text{otherwise.} \end{cases})$$

Adequacy: $\omega\mathbf{Qbs}$ is an adequate model for IpPCF using the interval integration monad $\mathbf{I}T$.

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Push-forward of α along Leb is

$$\begin{aligned} (\alpha_* \mathrm{Leb})(A) &= \mathrm{Leb}(\{r \in \mathbb{R} \mid \alpha(r) \in A\}) \\ &= \begin{cases} \mathrm{Leb}((0,1)) = 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases} \\ &= \delta_x(A) \end{aligned}$$

Adequacy: ω**Qbs** is an adequate model for IpPCF using the interval integration monad $\mathbf{I}T$.

Step 1: $[\![\sigma]\!]$ is sharp

Recall our results

1.1 $[\![\mathfrak{R}]\!]$ is ω-continuous.

1.2 $\omega\mathbf{Qbs}(X, Y)$ is bounded-complete, pointed and sharp, if X and Y are bounded-complete, pointed and sharp.

1.3 $\mathbf{I}TX$ is bounded-complete, pointed and **sharp**, if X is sharp.

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For any type σ, $[\![\sigma]\!]$ is bounded-complete, pointed and sharp.

- $[\![\mathfrak{R}]\!]_{\mathsf{I}} := (\mathsf{I}\mathbb{R}, M_{\mathsf{I}\mathbb{R}}, \sqsubseteq)_{\perp}$ is bounded-complete, pointed and sharp.
- $[\![\mathfrak{B}]\!]_{\mathsf{I}} := (\{\mathrm{tt}, \mathrm{ff}\}, M, =)_{\perp}$ is bounded-complete, pointed and sharp.
- $[\![\sigma \Rightarrow \tau]\!]_{\mathsf{I}} := [\![\sigma]\!]_{\mathsf{I}} \Rightarrow \mathsf{I}T[\![\tau]\!]_{\mathsf{I}}$ is bounded-complete, pointed and sharp if $[\![\sigma]\!]_{\mathsf{I}}$ and $[\![\tau]\!]_{\mathsf{I}}$ are also.

Is $\omega\mathbf{Qbs}$ a fully abstract model for IpPCF?

Full Abstraction

$[\![s]\!] = [\![t]\!]$ if and only if $s \sim_{ctx} t$.

Two Steps to Full Abstraction

1. For any type σ, $[\![\sigma]\!]$ is sharp. ✓
   - 1.1 $[\![\mathfrak{R}]\!]$ is bounded-complete, pointed and sharp. ✓
   - 1.2 $\omega\mathbf{Qbs}(X, Y)$ is bounded-complete, pointed and sharp, if X and Y are bounded-complete, pointed and sharp. ✓
   - 1.3 $\mathbf{I}TX$ is bounded-complete, pointed and **sharp with basis** $\{\bigsqcup_{i=1}^{n} \eta_i \mid \eta_i \in \oint B(\mathbf{I}\mathbb{R} \Rightarrow X_\perp) \wedge n > 0\}$, if X is sharp. ✓
2. Show that all basis elements of $[\![\sigma]\!]$ are definable.

Step 2: Are all basis elements of $[\![\sigma]\!]$ definable?

Proposition

If all elements of $B(\mathbf{I}\mathbb{R} \Rightarrow [\![\sigma]\!])$ are definable, then all elements in $\oint B(\mathbf{I}\mathbb{R} \Rightarrow [\![\sigma]\!])$ are definable. (Used the proof idea in the definability result in Staton 2017)

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Conjecture

If IpPCF is extended with parallel function symbols and supremum operator, then all basis elements of $[\![\sigma]\!]$ are definable.

- CBV PCF extended with parallel-if, lazy PCF and PCF with control are fully abstract. (Sieber 1990)
- Real PCF extended with sup is universal and hence fully abstract, where $\sup_{[a,b]} f := \bigsqcup_{P \in \mathfrak{P}[a,b]} \max_{x \in P} f(x)$. (Edalat and Escardó 1996)

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Summary and Future Work

Syntax	Prob PCF with Partial Real Numbers (IpPCF)
Semantics	Using $\omega\mathbf{Qbs}$ and Interval Integration Monad $\mathbf{I}T$
Correctness ✓	$s \longrightarrow_n \oint \alpha$ implies $[\![s]\!] = \lambda f.\int_{\Lambda \vdash \sigma} [\![t]\!] f \ (\alpha_* \mathrm{Leb})(dt)$
Adequacy ✓	$[\![s]\!] = [\![t]\!]$ implies $s \sim_{ctx} t$

Is $\omega\mathbf{Qbs}$ a fully abstract model for IpPCF?

1. For any type σ, $[\![\sigma]\!]$ is sharp. ✓
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